A FLUID-DYNAMIC MODEL FOR TRAFFIC FLOW ON A ROAD NETWORK

1. Traffic Flows

Here we refer to the book of Haberman [2] and to the papers [1, 3] for the description of the mathematical model of traffic flows on road networks. The study of traffic problems proposes to answer to several questions: where to install traffic lights or stop signs; how long the cycle of traffic lights should be; where to construct entrances, exits, and overpasses. The principal aim is to discover traffic phenomena in order to eventually take decisions which may alleviate congestion, maximize flow of traffic, eliminate accidents, and so on. Here we focus our attention on the traffic flow along a unidirectional highway. We will analyze traffic situations resulting from the complex interaction of many vehicles, instead of studying the behavior of individual cars. Here we will only formulate deterministic mathematical models, but it is also possible to develop statistical theories. The treatment of these problems is based on the fundamental traffic variables: velocity, density and flow. The nonlinear partial differential equation

$$\partial_t \rho + \partial_x f(\rho) = 0 \tag{1.1}$$

is the consequence of conservation of cars and experimental relationships between car velocity and traffic density.

2. Velocity Field

Let us consider a car moving along a highway. There are two ways to measure velocity. The most common is to record the velocity $v_i = \frac{dx_i}{dt}$ of each car. With N cars there are different velocities, $v_i(t), i = 1, \dots, N$, each depending on time. If the number of cars N is large, it becomes difficult to keep track of each car. So, instead of measuring the velocity of each individual car, we associate to each point in space at each time a **velocity field**, $v(x, t)$. This would be the velocity measured by an observer fixed at position x at time t.

3. Traffic Flow and Traffic Density

In addition to car velocities, an observer fixed at a certain position along the highway, could measure the number of cars that passed in a given length of time. The average number of cars passing per time

unit (for example one minute) is called the **traffic flow** $q = q(x,t)$.
A systematic procedure could be used to take into account cars completely in a given region at a fixed time; estimates of fractional cars could be used or a car could be counted only if its center is in the region. These measurements give the **density** of cars, ρ, that represents the number of cars per distance unit (for example hundred of meters).

4. Flow equals density times velocity

There is a close relationship between the three fundamental traffic variables: velocity, density and flow. It is quite realistic to think to the flux q - the number of cars per time unit - as a function of the only density ρ. More precisely the flux will be expressed as

$$\boxed{q(x,t) = \rho(x,t)v(x,t)} \tag{4.1}$$

that means

$$\boxed{\text{traffic flow} = (\text{traffic density}) \times (\text{mean velocity})}$$

As the density increases (meaning there are more and more cars per meter), the velocity of cars diminishes. Thus we make the hypotesis that the velocity of cars at any point of the road is a regular strictly decreasing function of the density:

$$\boxed{v = v(\rho).}$$

Lighthill and Whitham and independently Richards in the mid-1950s proposed this type of mathematical model of traffic flow.
If there are no other cars on the highway (corresponding to very low traffic densities), then the car would travel at the maximum speed v_{max},

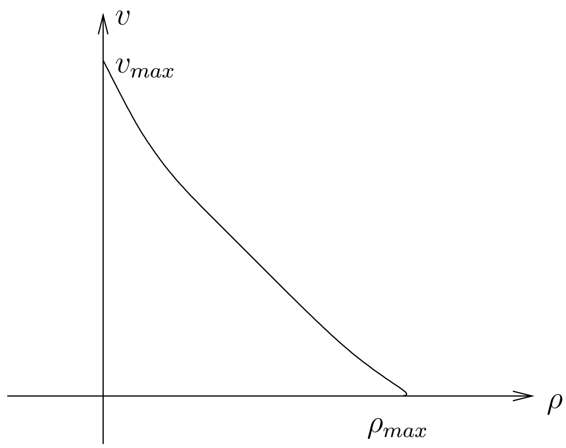
$$\boxed{v(0) = v_{max}.}$$

v_{max} is sometimes referred to as the "mean free speed" corresponding to the velocity cars would travel if they were free from interference from other cars. At a certain density cars stop before they touch to each other. This maximum density, ρ_{max}, usually corresponds to what is called bumper-to-bumper traffic:

$$\boxed{v(\rho_{max}) = 0.}$$

4.1. **Conservation of the number of cars.**

Let us fix a certain segment (a,b) on the highway and two quite close times $t_1 < t_2$. We are assuming that no cars are created or destroyed in the interval, then the changes in the number of cars result from crossings at $x = a$ and $x = b$ only. We deduce that the cars entered



from the point a at a certain time will exit from the point b. Thus the difference of the the total quantity of cars in the segment between the two considered instants

$$\int_a^b \rho(x,t_2)dx - \int_a^b \rho(x,t_1)dx$$

must be equal to the difference of the total flux at the endpoints

$$\int_{t_1}^{t_2} q(a,t)dt - \int_{t_1}^{t_2} q(b,t)dt.$$

Dividing the integrals for the product of $b-a$ and t_2-t_1 and taking the limits $(b-a)\to 0$ and $(t_2-t_1)\to 0$, with the assumption that v and q are regular, we finally obtain the conservation law:

$$\rho_t + q_x = 0. \tag{4.2}$$

Taking the velocity as

$$v(\rho) = v_{max}\left(1 - \frac{\rho}{\rho_{max}}\right),$$

we have the flux

$$q(\rho) = v_{max}\left(1 - \frac{\rho}{\rho_{max}}\right)\rho.$$

The flux is null if there are no cars or if the density is maximum and it reaches the maximum for $\rho = \frac{\rho_{max}}{2}$. It is easy to see the presence of discontinuity if someone brakes. The density assumes a discontinuity that propagates backwards along the queue.
For further details see [2].

References

[1] M. J. Lighthill, G. B. Whitham. On kinematic waves. II. A theory of traffic flow on long crowded roads. *Proc. Roy. Soc. London. Ser. A.*, **229** (1955), 317–345.

[2] R. Haberman. Mathematical models. Prentice-Hall, Inc. New Jersey, 1977, 255-394.

[3] P. I. Richards. Shock Waves on the Highway. *Oper. Res.*, **4** (1956), 42-51.