# A FLUID-DYNAMIC MODEL AT THE JUNCTIONS

## 1. Basic definitions for road networks

For the notions about the model given in the sequel we refer to the paper by Coclite-Piccoli-Garavello [2].
Different types of mathematical models are used for the simulation of vehicular traffic. They can be roughly classified in microscopic, mesoscopic and macroscopic. The basic models are the car following or microscopic models based on Newton's law. The macroscopic models seem to properly treat some phenomena such as shocks creation and propagation. Here we propose a fluid-dynamic model for traffic flow on a road network, which can be applied to the case of crossings with lights and circles. We consider the conservation law formulation proposed by Lighthill-Whitham and Richards. More precisely, one considers the conservation of cars described by the equation

$$\partial_t \rho + \partial_x f(\rho) = 0, \tag{1.1}$$

where $\rho = \rho(x,t)$ is the density of cars, with $\rho \in [0, \rho_{max}]$, $(x,t) \in \mathbb{R}^2$ and $\rho_{max}$ is the maximum density of cars on the road; $f(\rho)$ is the flux, which can be written $f(\rho) = \rho v(\rho)$, with $v(x,t)$ the velocity. Tipically $v$ is assumed to be a smooth decreasing function of $\rho$.
Here we are interested in a road network. This means that we have a finite number of roads modelled by intervals $[a_i, b_i]$ (with one of the endpoints eventually infinite) that meet at the some junctions. We give boundary data and solve the associated boundary problem for the endpoints (not infinite) that do not meet at any junction. Junctions play a fundamental role, as the system at a junction is underdetermined, even after prescribing the conservation of cars. The Rankine-Hugoniot at a junction reads:

$$\sum_{i=1}^{n} f(\rho_i(t, b_i)) = \sum_{j=n+1}^{n+m} f(\rho_j(t, a_j)),$$

where $\rho_i$, $i = 1, \dots, n$, are the car densities on incoming roads; $\rho_j$, $j = n+1, \dots, n+m$, are the car densities on the outgoing roads.
To determine a unique solution to Riemann problems at junctions, assume the following criteria:

- **(A):** there are some fixed coefficients, the prescribed preferences of drivers, that express the distribution of traffic from incoming to outgoing roads;

**(B):** respecting (A), drivers choices are made in order to maximize the flux.

Let us consider the rule (A). We fix a matrix, called **traffic distribution matrix**:

$$A = \{\alpha_{ji}\}_{j=n+1,\dots,n+m, i=1,\dots,n} \in \mathbb{R}^{m\times n},$$

such that

$$0 < \alpha_{ji} < 1, \quad \sum_{j=n+1}^{n+m} \alpha_{ji} = 1, \tag{1.2}$$

for $i = 1,\dots,n$ and $j = n+1,\dots,n+m$, where $\alpha_{ji}$ is the percentage of drivers arriving from the $i$-th incoming road that take the $j$-th outgoing road.

**Remark 1.1** Note that the only the rule (A) is not sufficient to have a unique solution to Riemann problems, that are still under-determined.

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Under suitable assumptions on $A$ and rules (A)-(B), representing a situation where drivers have a final destination and maximize the flux whenever is possible, Riemann problems can be uniquely solved. In [2] it has been proved existence of each solution to Cauchy problems respecting rules (A) and (B).

It is possible to introduce time dependent coefficients for the rule (A), and in particular traffic lights are modeled to deal by periodic coefficients. In the same way, we can treat networks assigning a different flux function $f_i$ on each road $I_i$.

Let us first recall the basic definitions and results from [2]. The parametrization of roads composing a network is made through a set of intervals $I_i = [a_i, b_i] \subset \mathbb{R}$, $i \in 1,\dots,N$, with the endpoints possibly infinite. The datum is a finite collection of densities $\rho_i$ defined on $I_i \times [0,+\infty)$.

$\rho_i$ is a weak entropy solution on road $I_i$, if for every $\varphi : I_i \to \mathbb{R}$ smooth and with compact support on $(a_i, b_i) \times (0,+\infty)$ one has

$$\int_{a_i}^{b_i}\int_0^{+\infty}\left(\rho_i\frac{\partial\varphi}{\partial t} + f(\rho_i)\frac{\partial\varphi}{\partial x}\right) dx\ dt = 0 \tag{1.3}$$

and for every $k \in \mathbb{R}$ and $\tilde{\varphi} : I_i \to \mathbb{R}$ smooth, positive with compact support on $(a_i, b_i) \times (0,+\infty)$

$$\int_{a_i}^{b_i}\int_0^{+\infty}\left(|\rho_i - k|\frac{\partial\tilde{\varphi}}{\partial t} + sgn(\rho_i - k)(f(\rho_i) - f(k))\frac{\partial\tilde{\varphi}}{\partial x}\right) dx\ dt \geq 0 \tag{1.4}$$

For equation (1.1) on $\mathbb{R}$ it is well-known that there exists a unique weak entropy solution for every initial data belonging to $L^\infty$, with a continuous dependence on the initial data in $L^1_{loc}$. Roads are linked to each other by some junctions, with the assumption that each road can be incoming at most for one junction and outgoing at most for one junction. Consequently the complete model is given by a pair $(\mathcal{I}, \mathcal{J})$, with $\mathcal{I} = \{I_i : i = 1, \ldots, N\}$ the collection of roads and $\mathcal{J}$ the number of junctions.

Consider a junction $J$ with $n$ incoming roads, say $I_1, \ldots, I_n$, and $m$ outgoing roads, say $I_{n+1}, \ldots, I_{n+m}$. A *weak solution at the junction $J$* is a collection of functions $\rho_l : [0, +\infty[\times I_l \to \mathbb{R}$, $l = 1, \ldots, n+m$, such that

$$\sum_{l=0}^{n+m} \left( \int_0^{+\infty} \int_{a_l}^{b_l} \left( \rho_l \frac{\partial \varphi_l}{\partial t} + f(\rho_l) \frac{\partial \varphi_l}{\partial x} \right) dx\ dt \right) = 0, \tag{1.5}$$

for every $\varphi_l$, $l = 1, \ldots, n+m$, smooth having compact support in $(0, +\infty) \times (a_l, b_l]$ for $l = 1, \ldots, n$ (incoming roads) and in $(0, +\infty) \times [a_l, b_l)$ for $l = n+1, \ldots, n+m$ (outgoing roads), that are also *smooth across the junction*, i.e.

$$\varphi_i(\cdot, b_i) = \varphi_j(\cdot, a_j), \qquad \frac{\partial \varphi_i}{\partial x}(\cdot, b_i) = \frac{\partial \varphi_j}{\partial x}(\cdot, a_j), \qquad i = 1, ..., n, j = n+1, ..., n+m.$$

**Remark 1.2** Let $\rho = (\rho_1, \ldots, \rho_{n+m})$ be a weak solution at the junction such that each $x \to \rho_i(t, x)$ has bounded variation. We can deduce that $\rho$ satisfies the *Rankine-Hugoniot Condition* at the junction $J$, namely

$$\sum_{i=1}^{n} f(\rho_i(t, b_i-)) = \sum_{j=n+1}^{n+m} f(\rho_j(t, a_j+)), \tag{1.6}$$

for almost every $t > 0$.

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The rules (A) and (B) can be given explicitly only for solutions with bounded variation as in the next definition.

**Definition 1.3** *Let $\rho = (\rho_1, \ldots, \rho_{n+m})$ be such that $\rho_i(x, t)$ is of bounded variation for every $t \geq 0$. Then $\rho$ is an admissible weak solution of (1.1) associated to the matrix $A$, satisfying (1.2), at the junction $J$ the following properties hold:*

*(i) $\rho$ is a weak solution at the junction;*
*(ii) $f(\rho_j(\cdot, a_j^+)) = \sum_{i=1}^{n} \alpha_{ji} f(\rho_i(\cdot, b_i^+))$, for $j = n+1, \ldots, n+m$;*
*(iii) $f(\rho_i(\cdot, b_i^-)) + \sum_{j=n+1}^{n+m} f(\rho_j(\cdot, a_j^+))$, is maximum subject to (ii).*

A boundary data $\psi_i : [0, +\infty] \to \mathbb{R}$ is assigned in the following cases: for each road $I_i = [a_i, b_i]$, if $a_i > -\infty$ and $I_i$ is not the outgoing road of any junction, or if $b_i < +\infty$ and $I_i$ is not the incoming road of any junction. If boundary data is given, we need $\phi_i$ to verify $\rho_i(t, a_i) = \psi_i(t)$ or $\rho_i(t, b_i) = \psi_i(t)$ in the sense of [1].

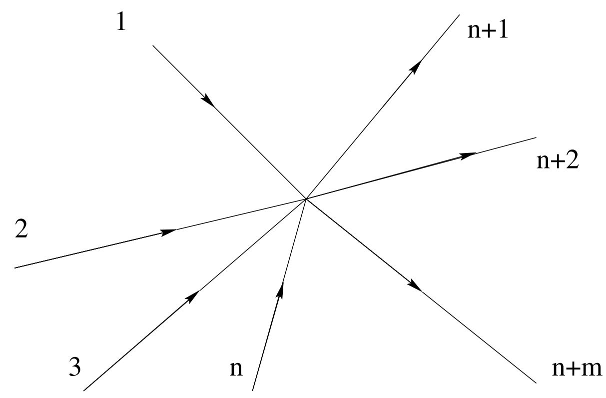


FIGURE 1. junction.

**Definition 1.4** *Given $\bar{\rho}_i : I_i \to \mathbb{R}$ and possibly $\psi_i : [0,+\infty[\to \mathbb{R}$, functions of $L^\infty$, a collection of functions $\rho = (\rho_1, \ldots, \rho_N)$ with $\rho_i : [0,+\infty[\times I_i \to \mathbb{R}$ continuous as functions from $[0,+\infty[$ into $L^1_{loc}$, is an admissible solution if $\rho_i$ is a weak entropic solution to (1.1) on $I_i$, $\rho_i(0,x) = \bar{\rho}_i(x)$ a.e., $\rho_i(t,b_i) = \psi_i(t)$ in the sense of* [1]*, finally such that at each junction $\rho$ is a weak solution and is an admissible weak solution in case of bounded variation.*

We recall the construction of solutions to the Riemann problems for rules (A) and (B). A Riemann problem for a scalar conservation law is a Cauchy problem for an initial data of Heaviside type, that is piecewise constant with only one discontinuity. Once Riemann problems are solved, a solution to Cauchy problems can be obtained, for instance, by wave front tracking. In case of concave or convex fluxes, the Riemann solutions are of two types: continuous waves called rarefactions and travelling discontinuity called shocks. The speed of the waves is related to $f'(\rho)$.

For a junction, as for a scalar conservation law, a Riemann problem is a Cauchy problem with an initial data that is constant on each road. Let us make the subsequent assumptions on the flux:

- **($\mathcal{F}$)** $f : [0,1] \to \mathbb{R}$ is smooth, strictly concave (i.e. $f'' \leq -c < 0$ for some $c > 0$), $f(0) = f(1) = 0$, $|f'(x)| \leq C < +\infty$. Hence there exists a unique $\sigma \in ]0,1[$ such that $f'(\sigma) = 0$ (that is $\sigma$ is a strict maximum).

Consider a junction $J$ with $n$ incoming roads and $m$ outgoing roads. The densities of the cars on the incoming roads are indicated by:

$$(x,t) \in \mathbb{R}^+ \times I_i \mapsto \rho_i(x,t) \in [0,1], \quad i \in \{1, \ldots, n\}$$

and those on the outgoing roads:

$$(x,t) \in \mathbb{R}^+ \times I_j \mapsto \rho_j(x,t) \in [0,1], \quad j \in \{1, \ldots, m\}.$$

We introduce the following application:

**Definition 1.5** *Let $\tau : [0,1] \mapsto [0,1], \tau(\sigma) = \sigma$, be the map satisfying the following*

$$\tau(\rho) \neq \rho, \qquad f(\tau(\rho)) = f(\rho),$$

*for each $\rho \neq \sigma$.*
*Evidently $\tau$ is well-defined and it verifies*

$$0 \leq \rho \leq \sigma \Longleftrightarrow \sigma \leq \tau(\rho) \leq 1, \qquad \sigma \leq \rho \leq 1 \Longleftrightarrow 0 \leq \tau(\rho) \leq \sigma.$$

In order to ensure uniqueness of the solution to Riemann problems we need some generic additional conditions on the matrix $A$. Let $\{e_1, \ldots, e_n\}$ be the canonical basis of $\mathbb{R}^n$ and for every subset $V \subset \mathbb{R}^n$, indicate by $V^\perp$ its orthogonal. For every $i = 1, \ldots, n$, let us define $H_i$ the coordinate hyperplane orthogonal to $e_i$ and for every $j = n+1, \ldots, n+m$ define $H_j = {\alpha_j}^\perp$, with $\alpha_j = (\alpha_{j1}, \ldots, \alpha_{jn})$. Indicate by $\mathcal{K}$ the set of indices $k = (k_1, \ldots, k_l)$, $1 \leq l \leq n-1$, such that $0 \leq k_1 < k_2 < \cdots < k_l \leq n+m$ and for every $k \in \mathcal{K}$ we set

$$H_k = \bigcap_{h=1}^{l} H_{k_h}.$$

Letting $\mathbf{1} = (\mathbf{1}, \ldots, \mathbf{1}) \in \mathbb{R}^{\mathbf{n}}$, we assume

$(RP)$ for every $k \in \mathcal{K}$, $\mathbf{1} \notin \mathbf{H}_{\mathbf{k}}^{\perp}$.

From $(RP)$ easily follows $m \geq n$, for details see [2].

The existence and uniqueness of admissible solutions for the Riemann problem of a junction is expressed by the following theorem.

**Theorem 1.6** *Let $f : [0,1] \to \mathbb{R}$ satisfy $(\mathcal{F})$, the matrix $A$ satisfy $(C)$ and $\rho_{1,0}, \ldots, \rho_{n+m,0} \in [0,1]$ be constants. There exists a unique admissible weak solution, in the sense of Definition 1.3, namely $\rho = (\rho_1, \ldots, \rho_{n+m})$ of (1.1) at the junction $J$ such that*

$$\rho_1(0, \cdot) \equiv \rho_{1,0}, \ldots, \rho_{n+m}(0, \cdot) \equiv \rho_{n+m,0}$$

*Moreover, there exists a unique $(n+m)$–uple $(\hat{\rho}_1, \ldots, \hat{\rho}_{n+m}) \in [0,1]^{n+m}$, such that*

$$(1.7) \quad \hat{\rho}_i \in \begin{cases} \{\rho_{i,0}\} \cup (\tau(\rho_{i,0}), 1] & \text{if} \quad 0 \leq \rho_{i,0} \leq \sigma, \\ [\sigma, 1] & \text{if} \quad \sigma \leq \rho_{i,0} \leq 1, \end{cases} \qquad i = 1, \ldots, n$$

*and,*
(1.8)

$$\hat{\rho}_j \in \begin{cases} [0, \sigma] & \text{if} \quad 0 \leq \rho_{j,0} \leq \sigma, \\ \{\rho_{j,0}\} \cup [0, \tau(\rho_{j,0})) & \text{if} \quad \sigma \leq \rho_{j,0} \leq 1. \end{cases} \qquad j = n+1, \ldots, n+m.$$

*Fixed* $i \in \{1,\ldots,n\}$*, if* $\rho_{i,0} \le \hat{\rho}_i$ *the solution is a shock:*

$$\rho_i(x,t) = \begin{cases} \rho_{i0} & \textit{if } x \le \frac{f(\hat{\rho}_i)-f(\rho_{i,0})}{\hat{\rho}_i-\rho_{i,0}}t, \\ \\ \hat{\rho}_i & \textit{otherwise,} \end{cases} \tag{1.9}$$

*and if* $\rho_{i,0} > \hat{\rho}_i$ *the solution is a rarefaction:*

$$\rho_i(x,t) = \begin{cases} \rho_{i0} & \textit{if } x \le f'(\rho_{i,0})t, \\ (f')^{-1}\left(\frac{x}{t}\right) & f'(\rho_{i,0})t \le x \le f'(\hat{\rho}_i)t, \\ \hat{\rho}_i & \textit{if } x > f'(\hat{\rho}_i)t. \end{cases} \tag{1.10}$$

*Proof.* Define the map

$$E : (\gamma_1, ..., \gamma_n) \in \mathbb{R}^n \longmapsto \sum_{i=1}^{n} \gamma_i$$

and the sets

$$\Omega_i \doteq \begin{cases} [0, f(\rho_{i,0})], & \text{if } 0 \le \rho_{i,0} \le \sigma, \\ [0, f(\sigma)], & \text{if } \sigma \le \rho_{i,0} \le 1, \end{cases} \qquad i = 1, ..., n,$$

$$\Omega_j \doteq \begin{cases} [0, f(\sigma)], & \text{if } 0 \le \rho_{j,0} \le \sigma, \\ [0, f(\rho_{j,0})], & \text{if } \sigma \le \rho_{j,0} \le 1, \end{cases} \qquad j = n+1, ..., n+m,$$

$$\Omega \doteq \left\{ (\gamma_1, ..., \gamma_n) \in \Omega_1 \times \ldots \times \Omega_n \middle| A \cdot (\gamma_1, ..., \gamma_n)^T \in \Omega_{n+1} \times \ldots \times \Omega_{n+m} \right\}.$$

The set $\Omega$ is closed, convex and not empty. Furthermore, by (RP), $\nabla E$ is not orthogonal to any nontrivial subspace contained in a supporting hyperplane of $\Omega$, therefore there exists a unique vector $(\hat{\gamma}_1, ..., \hat{\gamma}_n) \in \Omega$ such that

$$E(\hat{\gamma}_1, ..., \hat{\gamma}_n) = \max_{(\gamma_1,...,\gamma_n)\in\Omega} E(\gamma_1, ..., \gamma_n).$$

For every $i \in \{1, ..., n\}$, we choose $\hat{\rho}_i \in [0,1]$ such that

$$f(\hat{\rho}_i) = \hat{\gamma}_i, \quad \hat{\rho}_i \in \begin{cases} \{\rho_{i,0}\} \cup ]\tau(\rho_{i,0}), 1], & \text{if } 0 \le \rho_{i,0} \le \sigma, \\ [\sigma, 1], & \text{if } \sigma \le \rho_{i,0} \le 1. \end{cases} \tag{1.11}$$

By $(F)$, $\hat{\rho}_i$ exists and is unique. Let

$$\hat{\gamma}_j \doteq \sum_{i=1}^{n} \alpha_{ji} \hat{\gamma}_i, \qquad j = n+1, ..., n+m,$$

and $\hat{\rho}_j \in [0,1]$ be such that

$$f(\hat{\rho}_j) = \hat{\gamma}_j, \quad \hat{\rho}_j \in \begin{cases} [0, \sigma], & \text{if } 0 \le \rho_{j,0} \le \sigma, \\ \{\rho_{j,0}\} \cup [0, \tau(\rho_{j,0})[, & \text{if } \sigma \le \rho_{j,0} \le 1. \end{cases} \tag{1.12}$$

Since $(\hat{\gamma}_1, ..., \hat{\gamma}_n) \in \Omega$, $\hat{\rho}_j$ exists and is unique for every $j \in \{n+1, \ldots, n+m\}$. The thesis is achieved.

The solution on each road is given by the solution to Riemann problem with data $(\rho_{i0}, \hat{\rho}_i)$ for incoming roads and $(\hat{\rho}_j, \rho_{j0})$ for outgoing roads. Once the solution to Riemann problems is obtained, one can use a wave front tracking algorithm to build a sequence of approximate solutions.

**Remark 1.7** In order to have admissible solutions to Riemann problems, we need that $(\rho_{i0}, \hat{\rho}_i)$ is solved by waves with negative speed, while $(\hat{\rho}_j, \rho_{j0})$ is solved by waves with positive speed. This is equivalent to conditions (1.7) and (1.8).

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## 2. Existence of solutions.

Once the solution of Riemann problems at junctions is obtained, using that the speed of propagation is finite, one constructs solutions via wave-front tracking algorithm.
Now we are assuming to have junctions composed by two incoming and two outgoing roads. We are able to give an estimate of the total variation of the flux along an approximate wave front tracking solution.

**Lemma 2.1** *Consider a road network $(\mathcal{I}, \mathcal{J})$. For some $K > 0$ we have the estimate on the flux variation*

$$\begin{aligned} Tot.Var.(f(\rho(t,\cdot))) &\leq e^{Kt} Tot.Var.(f(\rho(0+,\cdot))) \\ &\leq e^{Kt} Tot.Var.(f(\rho(0,\cdot))) + 2Rf(\sigma) \end{aligned}$$

*for each $t \geq 0$, with $R$ the total number of roads of the network.*

Now we can state the existence result for the approximate solution.

**Theorem 2.2** *Fix a road network $(\mathcal{I}, \mathcal{J})$. Given $C > 0$ and $T > 0$, there exists an admissible solution defined on $[0, T]$ for every initial data $\bar{\rho} \in cl\{\rho : TV(\rho) \leq C\}$, where $cl$ is the closure in $L^1_{loc}$.*

For the proof of these results see again [2].

2.1. **Examples.**

**Example 2.3 (2 incoming - 2 outgoing roads)**
Here we consider the particular case of a junction with two outgoing

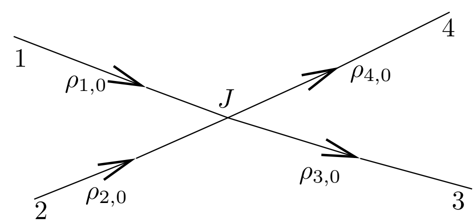


FIGURE 2. A junction with two incoming and two outgoing roads.

and two incoming roads. The incoming roads are indicated as 1 e 2, while the outgoing roads are 3 and 4.
In order to determine the region for the maximization of the flux, we impose a restriction on the initial data. For roads $i = 1, 2$ the maximum flux reads:

$$f_i^{max} = \begin{cases} f(\sigma) & \text{if} \quad \rho_{i,0} \in [\sigma, \rho_{max}], \\ \\ f(\rho_{i,0}) & \text{if} \quad \rho_{i,0} \in [0, \sigma), \end{cases}$$

while for roads $j = 3, 4$ the maximum flux is:

$$f_j^{max} = \begin{cases} f(\sigma) & \text{if } \rho_{j,0} \in [0, \sigma], \\ \\ f(\rho_{j,0}) & \text{if} \quad \rho_{j,0} \in (\sigma, \rho_{max}]. \end{cases}$$

We obtain the two sets:

$$\Omega_{12} = [0, f(\bar{\rho}_{10})] \times [0, f(\bar{\rho}_{20})],$$
$$\Omega_{34} = [0, f(\bar{\rho}_{30})] \times [0, f(\bar{\rho}_{40})],$$

and maximize the sum of fluxes on the region $\Omega_{12} \cap A^{-1}(\Omega_{34})$. Introducing the notation $\gamma_l = f(\bar{\rho}_{l,0})$, $l = 1, 2, 3, 4$, we have

$$\max(\gamma_1 + \gamma_2) = \hat{\gamma}_1 + \hat{\gamma}_2$$

and obtain $\hat{\gamma}_3$ and $\hat{\gamma}_4$, through the following relation

$$A\hat{\gamma} \in \Omega_{34}, \tag{2.1}$$

where the traffic distribution matrix reads

$$A = \begin{pmatrix} \alpha_{31} & \alpha_{32} \\ \alpha_{41} & \alpha_{42} \end{pmatrix}.$$

The solution is:

$$(\hat{\gamma}_1, \hat{\gamma}_2, \hat{\gamma}_3, \hat{\gamma}_4)$$

and the corresponding $\hat{\rho}_l$ are given by the inversion of:

$$f(\hat{\rho}_l) = \hat{\gamma}_l, \quad l = 1, \ldots, 4.$$

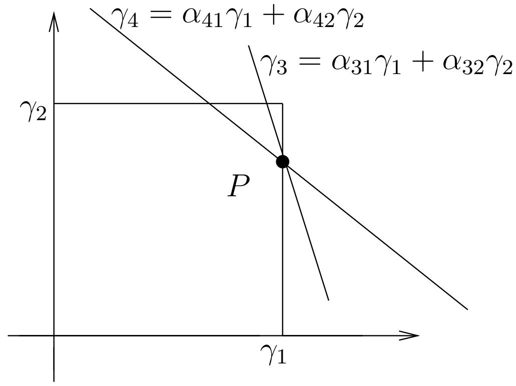


FIGURE 3. Maximization region.

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**Remark 2.4** We deduce that in order to treat the case $m < n$ it is necessary to introduce a further rule.

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**Example 2.5 (2 incoming - 1 outgoing roads)**
In particular, let us consider the case $m = 1, n = 2$. The two coefficients $\alpha_{31}$ and $\alpha_{32}$ must be equal to one. This is a fundamental mathematical issue, due to the fact that if not all cars can go through the junction, then there should be a yielding rule between incoming roads. To deal with this case we fix a new *right of way* parameter $q \in ]0, 1[$ and assign the rule:

**(C):** Assume that not all cars can enter the outgoing road and let $C$ be the quantity that can do it. Then $qC$ cars come from first incoming road and $(1 - q)C$ cars from the second.

The rule (C) allows to uniquely solve Riemann's problems.

In order to show how rule (C) works, let us consider a junction like the one showed in Fig. 4.

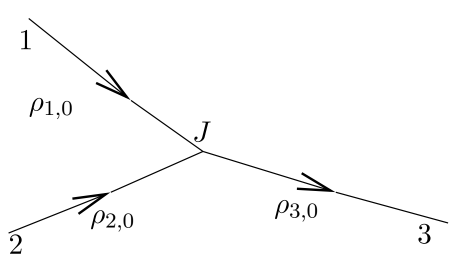


FIGURE 4. A junction with two incoming and one outgoing roads.

As explained before, condition (RP) on $A$ cannot hold for crossings with two incoming and one outgoing roads. Then we introduce a further parameter, whose meaning is the following. When the number of cars is too big to let all of them go through crossing, there is a yielding rule that describes the percentage of cars, going through the crossing, that comes from the first road.

Let us fix a crossing with two incoming roads $[a_i, b_i]$, $i = 1, 2$, and one outgoing road $[a_3, b_3]$ and assume that a right of way parameter $q \in ]0, 1[$ is given. The solution to the Riemann's problem $(\rho_{1,0}, \rho_{2,0}, \rho_{3,0})$ is composed by a single wave on each road connecting the initial states to $(\hat{\rho}_1, \hat{\rho}_2, \hat{\rho}_3)$ determined as follows (cfr. with the solution to the Riemann's problem in the two incoming two outgoing roads). Define $\gamma_i^{max}$, $i = 1, 2$ and $\gamma_3^{max}$ in the subsequent way:

$$\gamma_i^{max} = \begin{cases} f(\rho_{i,0}) & \text{if } \rho_{i,0} \in [0, \sigma], \\ f(\sigma) & \text{if } \rho_{i,0} \in ]\sigma, 1], \end{cases}$$

and

$$\gamma_3^{max} = \begin{cases} f(\sigma) & \text{if } \rho_{3,0} \in [0, \sigma], \\ f(\rho_{3,0}) & \text{if } \rho_{3,0} \in ]\sigma, 1]. \end{cases}$$

The quantities $\gamma_i^{max}$ represent the maximum flux that can be reached by a single wave solution on each road. Since our goal is to maximize going through traffic, we set:

$$\hat{\gamma}_3 = \min\{\gamma_1^{max} + \gamma_2^{max}, \gamma_3^{max}\}. \tag{2.2}$$

Consider the space $(\gamma_1, \gamma_2)$, then rule (C) is respected by points on the line:

$$\gamma_2 = \frac{1-q}{q}\gamma_1. \tag{2.3}$$

Thus define $P$ to be the point of intersection of the line (2.3) with the line $\gamma_1 + \gamma_2 = \hat{\gamma}_3$. Recall that the final fluxes should belong to the region:

$$\Omega = \{(\gamma_1, \gamma_2) : 0 \leq \gamma_i \leq \gamma_i^{max}\},$$

then we distinguish three cases:

a) $P$ is inside $\Omega$,
b) $P$ is outside $\Omega$,
c) $P$ is the upper-right vertex of $\Omega$ (that corresponds to the case $\hat{\gamma}_3 = \gamma_1^{max} + \gamma_2^{max}$).

In the first case we set $(\hat{\gamma}_1, \hat{\gamma}_2) = P$, while in the second we set $(\hat{\gamma}_1, \hat{\gamma}_2) = Q$, where $Q$ is the point of the segment $\Omega \cap \{(\gamma_1, \gamma_2) : \gamma_1 + \gamma_2 = \hat{\gamma}_3\}$ closest to the line (2.3). We show in Figure 5 the cases a)-b). In the third case, there is no need of using rule (C) and $(\hat{\gamma}_1, \hat{\gamma}_2) = P$, see Figure 6.

Then we determine $\hat{\rho}_i$ with the same rules of (1.7)-(1.8). The obtained solution is called the correct solution corresponding to parameter $q$.

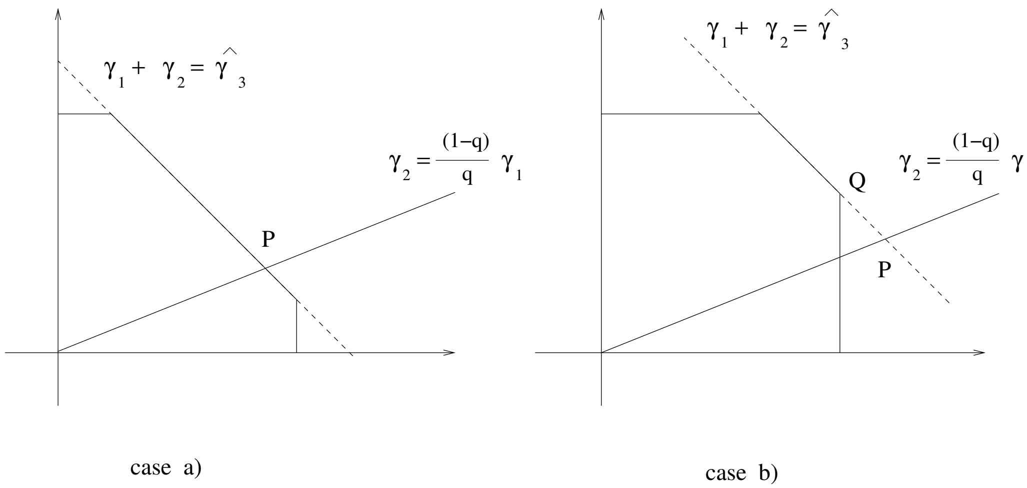


FIGURE 5. Solutions to Riemann's problem for rule (C).

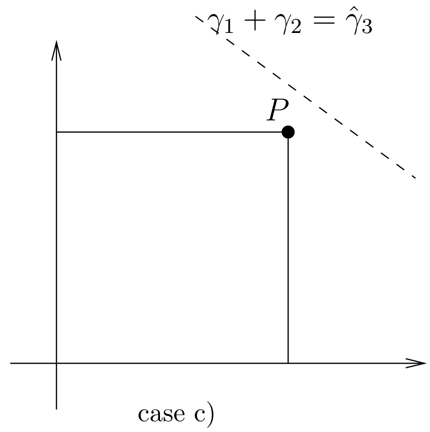


FIGURE 6. Solutions to Riemann's problem without using rule (C).

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**Example 2.6 (Bottleneck)**
The simplest application of the fluid-dynamic model presented in this Chapter is represented by the bottleneck, which is an example of a road with different fluxes.
In order to describe a bottleneck, it is necessary to consider two different flux functions along the road, where the conservation of cars is always expressed by (1.1) endowed with initial and boundary condition. For a bottleneck, in fact, we mean a road with different widths: in the first part of the street the flux reads

$$f_1(\rho) = \rho(1-\rho), \quad \rho \in [0,1], \tag{2.4}$$

while, in the narrowest part of the street, the flux considered is

$$f_2(\rho) = \rho\left(1 - \frac{3}{2}\rho\right), \quad \rho \in [0, 2/3]. \tag{2.5}$$

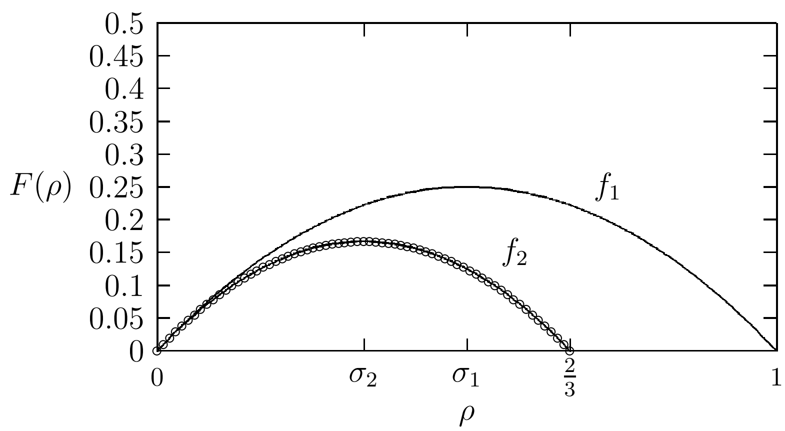


FIGURE 7. The flux functions $f_1(\rho)$ and $f_2(\rho)$.

As before, the maximum for the fluxes is unique:

$$f_1(\sigma_1) = \max_{[0,1]} f_1(\rho) = \frac{1}{4}, \quad \text{with } \sigma_1 = \frac{1}{2}, \tag{2.6}$$

$$f_2(\sigma_2) = \max_{[0,2/3]} f_2(\rho) = \frac{1}{6}, \quad \text{with } \sigma_2 = \frac{1}{3}. \tag{2.7}$$

A key role is played by the separation point between the two parts of the road, namely $B$. Indicate by $\rho_s$ the point placed on the left respect to $B$ (that belongs to the widest part of the street) and by $\rho_d$ the point of the narrowest part on the right respect to $B$ so that we can consider the road as composed by two different roads. The maximization of $f_1$ and $f_2$ is performed following the rules, respectively

$$f_1^{max}(\rho) = \begin{cases} f_1(\rho_s) & \text{if } \rho_s \leq \sigma_1, \\ f_1(\sigma_1) & \text{if } \rho_s \geq 1, \end{cases}$$

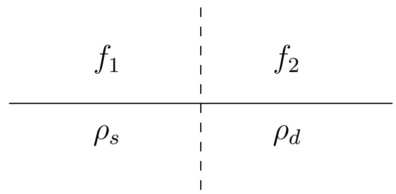


FIGURE 8. Interface at the bottleneck.

$$f_2^{max}(u) = \begin{cases} f_2(\sigma_2) & \text{if } \rho_d \leq \sigma_2, \\ f_2(\rho_d) & \text{if } \rho_d \geq \frac{2}{3} \end{cases}$$

and the intersection point between the two intervals is obtained taking the minimum

$$\gamma = \min\{f_1^{max}(\rho_s), f_2^{max}(\rho_d)\}, \tag{2.8}$$

with $\rho_s$ and $\rho_d$ instantaneously fixed. As the maximum density allowed in the second part is given by $\sigma_2 = \frac{1}{6}$, the creation of queues occurs when the density on the first road verifies

$$\rho(1-\rho) = \frac{1}{6} \Longleftrightarrow \bar{\rho} = \frac{1-\sqrt{\frac{1}{3}}}{2} \simeq 0.21\ . \tag{2.9}$$

Then, when $\rho_{1,b} < \bar{\rho}$ (recall that $\rho_{1,b}$ is the car density entering the largest road) there is no formation of shocks propagating backwards.

■

## 3. Traffic circles

Here we introduce the following traffic regulation problem: given a junction with some incoming roads and some outgoing ones, is it preferable to regulate the flux via a traffic light or via a traffic circle on which the incoming traffic enters continuously? More precisely, assuming that drivers arriving at the junction distribute on the outgoing roads according to some known coefficients our purpose is to understand which solution performs better from the point of view of total amount of cars going through the junction.

In order to treat this problem we need a model that describes the above situation and provides an accurate analysis. To this aim we consider the fluid dynamic model based on (1.1) and proposed in [2] adapted in a suitable way in order to treat the case of traffic circles.

3.1. **Solutions for traffic circles.** Recall the rule $(C)$ introduced in Section 1. $\hat{\rho}_i$ is determined with the same rules of (1.11)-(1.12) and the resulting solution is named the correct solution corresponding to parameter $q$.

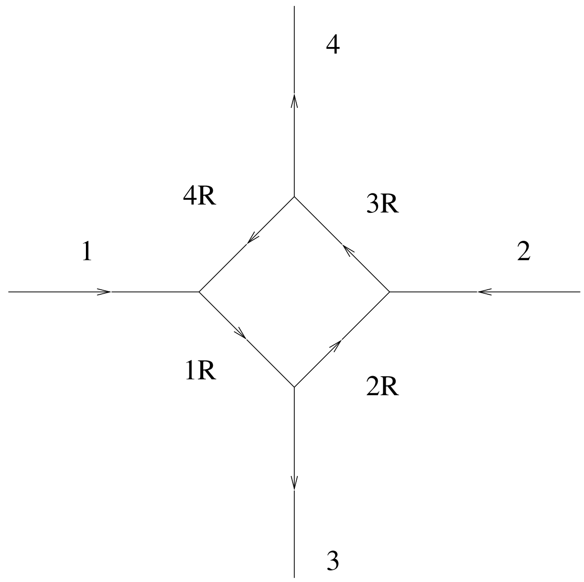


FIGURE 9. Traffic circle.

Consider a general network, as the traffic circle, with junctions having either one incoming and two outgoing or two incoming and one outgoing roads. Once the solution to Riemann's problems is fixed then we can introduce the definition of admissible solutions as in Definition 1.4. More precisely, given a set of parameters $q_k$ for all junctions $J_k$ with two incoming and one outgoing roads, a solution $\rho$ on the road network is admissible if for a.e. $t$ with $\rho(t)$ of bounded variation the Riemann's problem at each junction $J_k$ is solved in the correct way corresponding to the parameter $q_k$. With the same techniques of [3] one can construct admissible solutions.

In an entirely similar way we treat the case of coefficients $\alpha_{ij}$ and right of way parameters $q_k$ depending on time and having a finite number of discontinuities.
Notice that we only treat the case of the single-lane traffic circles. A model for the multi-lane traffic circles is proposed in [3].

3.1.1. *Low traffic.* Consider a simple network representing a traffic circle with a low traffic rate, in the sense that the number of cars reaching the circle is less then the capacity of the circle itself. There are four roads, named $1, \ldots, 4$, the first two incoming in the circle and the other two outgoing. In addition there are four roads $1R, \ldots, 4R$ that form the circle as in Figure 9.

As before the parameterization of roads is given by $[a_i, b_i]$, $i = 1, \ldots, 4$, and $[a_{iR}, b_{iR}]$, $i = 1, \ldots, 4$. We assign a traffic distribution matrix $A$ describing how traffic coming from roads $1, 2$ distributes through roads 3 and 4, passing by the intermediate roads of the circle. Two parameters are fixed, namely $\alpha, \beta \in ]0, 1[$, such that

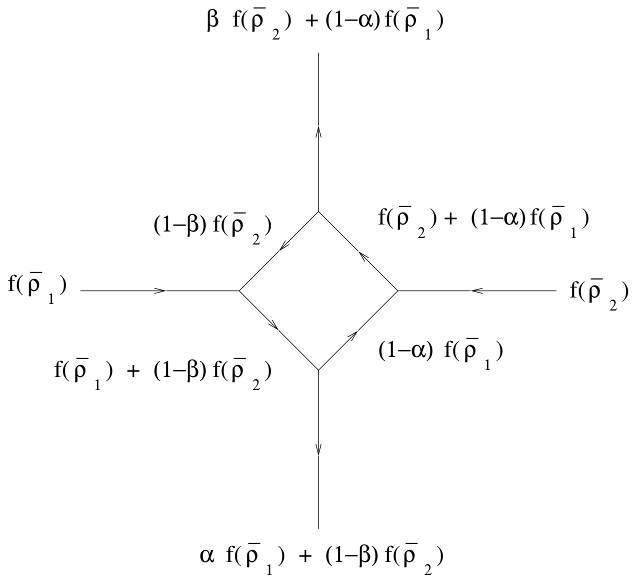


FIGURE 10. equilibrium for traffic circle

(C1) If $M$ cars reach the circle from road 1, then $\alpha M$ drive to road 3 and $(1-\alpha)M$ drive to road 4,

(C2) If $M$ cars reach the circle from road 2, then $\beta M$ drive to road 4 and $(1-\beta)M$ drive to road 3.

First we consider a static situation. We impose boundary conditions as follows

$$\rho_1(a_1,t) \equiv \bar{\rho}_1, \qquad \rho_2(a_2,t) \equiv \bar{\rho}_2, \tag{3.1}$$

with $\bar{\rho}_1$ and $\bar{\rho}_2$ constant fluxes from roads 1 and 2 respectively. Provided that roads 3 and 4 can absorb all incoming traffic, e.g. if

$$f(\bar{\rho}_1) + f(\bar{\rho}_2) \leq F(\sigma), \tag{3.2}$$

the situation of Figure 10 should be achieved. In particular, this happens with the following coefficients for the crossing (1R,3,2R)

$$\alpha_{1R,3} = \frac{\alpha f(\bar{\rho}_1) + (1-\beta) f(\bar{\rho}_2)}{f(\bar{\rho}_1) + (1-\beta) f(\bar{\rho}_2)}, \quad \alpha_{1R,2R} = \frac{(1-\alpha) f(\bar{\rho}_1)}{f(\bar{\rho}_1) + (1-\beta) f(\bar{\rho}_2)}, \tag{3.3}$$

and similarly for (3R,4,4R)

$$\alpha_{3R,4} = \frac{(1-\alpha) f(\bar{\rho}_1) + \beta f(\bar{\rho}_2)}{(1-\alpha) f(\bar{\rho}_1) + f(\bar{\rho}_2)}, \quad \alpha_{3R,4R} = \frac{(1-\beta) f(\bar{\rho}_2)}{(1-\alpha) f(\bar{\rho}_1) + f(\bar{\rho}_2)}. \tag{3.4}$$

Starting with an empty net, (3.2) is verified and the boundary data are given by (3.1). Then firstly the cars from road 1 and 2 reach road 3 and 4 respectively and the coefficients should be simply set as:

$$\alpha_{1R,3} = \alpha, \qquad \alpha_{1R,2R} = (1-\alpha), \qquad \alpha_{3R,4} = \beta, \qquad \alpha_{3R,4R} = (1-\beta). \tag{3.5}$$

Successively, since also cars from road 2 reach road 3 (and cars from road 1 reach road 4), we should modify in time the coefficients and finally set them as in (3.3) and (3.4). Due to this choice, there exists $T > 0$ such that the solution is given by the fluxes indicated in Figure 10 for every $t \geq T$, thus we can see that the problem is modeled suitably at not too heavy traffic level(corresponding to (3.2)). However, it is necessary to let the coefficients $\alpha$ vary on time, as specified by the subsequent theorem.

**Theorem 3.1** *Consider the circle network and assume (3.1), (3.2). There exists time dependent coefficients* $\alpha : [0, +\infty) \to [0, 1]$, *with (3.5) holding at time* 0 *and (3.3), (3.4) for large enough times, and* $T > 0$ *such that the solution* $\rho(t)$ *is constantly equal to that of Figure 10 for every* $t \geq T$.

For the proof see [3].

3.1.2. *Heavy traffic.* In this case the condition (3.2) is violated. More precisely, traffic jams is possible under one of the following conditions

$$f(\bar{\rho}_1) + (1 - \beta) f(\bar{\rho}_2) > f(\sigma), \tag{3.6}$$

$$(1 - \alpha) f(\bar{\rho}_1) + f(\bar{\rho}_2) > f(\sigma). \tag{3.7}$$

Consider situation of the traffic equilibrium for low traffic (Figure 10) but now with conditions (3.6)-(3.7) holding true. Shocks are then produced on some of roads at junctions (1,4R,1R) and (2,2R,3R). Observe that if one starts from empty circle then rarefaction waves start to fill up the circle approaching at some point a situation as in Figure 10.

For simplicity, from now on assume the following notation: each wave is indicated by $(f_l, f_r)$ where $f_l$ is the value of the flux to the left of the wave and $f_r$ the value of the flux to the right, being clear from the context which are the values of $\rho$ on the left and right of the wave. Set for simplicity:

$$f_1 := f(\bar{\rho}_1), \qquad f_2 := f(\bar{\rho}_2),$$

$$q_1 := q_{(1,4R,1R)}, \qquad q_2 := q_{(2,2R,3R)}.$$

For the junction (1,4R,1R) we have

$$\gamma_1^{max} = f_1,\ \gamma_{4R}^{max} = (1 - \beta) f_2\ , \gamma_{1R}^{max} = f(\sigma) = 1.$$

Then we have $\hat{\gamma}_{1R} = f(\sigma) = 1$, thus $\hat{\rho}_{1R} = \sigma$. Depending on the value of $q_1$ there are three cases:

(a) $q_1 \leq 1 - \frac{(1-\beta) f_2}{f(\sigma)}$, then $\hat{\gamma}_1 = f(\sigma) - (1 - \beta) f_2$ and $\hat{\gamma}_{4R} = (1 - \beta) f_2$;
(b) $1 - \frac{(1-\beta) f_2}{f(\sigma)} < q_1 < \frac{f_1}{f(\sigma)}$, then $\hat{\gamma}_1 = q\, f(\sigma)$ and $\hat{\gamma}_{4R} = (1 - q)\, f(\sigma)$;
(c) $q_1 \geq \frac{F_1}{f(\sigma)}$, then $\hat{\gamma}_1 = f_1$ and $\hat{\gamma}_{4R} = f(\sigma) - f_1$.

In case (a) a shock is produced on road 1 and no wave on road $4R$, in case (c) a shock is produced on road $4R$ and no wave on road 1, finally in case (b) a shock is produced on both roads.
An analogous analysis can be done for junction (2,2R,3R). First we put in case (a) for both junctions (1,4R,1R) and (2,2R, 3R), so that rarefactions are generated on roads 1R and 3R.
Note that the flux $f(\sigma)$ on road 2R is composed of $f(\sigma) - (1-\beta)f_2$ from road 1 and $(1-\beta)f_2$ from road 4R. As $\alpha$ of flux from road 1 comes out through road 3 and all flux from 4R comes out through road 3, we find that $\alpha f(\sigma) + (1-\alpha)(1-\beta)f_2$ exits road 3 and $(1-\alpha)(f(\sigma) - (1-\beta)f_2)$ goes to 2R. Condition (3.6) implies that $(1-\alpha)f_1 > (1-\alpha)(f(\sigma) - (1-\beta)f_2)$, thus on road 2R it produces a rarefaction which approaches the crossing (2,2R,3R). This rarefaction reduces the traffic flux entering from road 2R thus the circle is not stuck. For roads 3R and 4R the analysis is very similar.

**Proposition 3.2** *The traffic on the circle never gets stuck if the following holds:*

$$q_1 \leq 1 - \frac{(1-\beta)f_2}{f(\sigma)}, \qquad q_2 \leq 1 - \frac{(1-\alpha)f_1}{f(\sigma)}.$$

Suppose to be in case (b) for both junctions (1,4R,1R) and (2,2R, 3R). Then rarefactions are produced on roads 1R and 3R and shocks on the other roads.
The flux $f(\sigma)$ on road 2R is formed of $q_1 f(\sigma)$ from road 1 and $(1-q_1)f(\sigma)$ from road 4R. Since $\alpha$ of flux from road 1 gets out from road 3 and all flux from 4R gets out from road 3, one has that $(1-(1-\alpha)q_1)f(\sigma)$ exits road 3 and $(1-\alpha)q_1 f(\sigma)$ proceeds to 2R. That is to say

$$\alpha_{1R,2R} = (1-\alpha)q_1.$$

Then when the shock created on road 2R reaches the junction (2,2R,3R), then a shock is produced on 1R:

$$\left( f(\sigma), \frac{(1-q_2)f(\sigma)}{(1-\alpha)q_1} \right).$$

We can proceed analogously for roads 3R and 4R. Then we conclude that shocks are produced on the whole circle recursively. Let us introduce some more notation:

- $x_1^n$ is the value of the flux on road 1R after the $n$-th shock–junction interaction;
- $x_2^n$ is the value of the flux on road 2R after the $n$-th shock–junction interaction;
- $x_3^n$ is the value of the flux on road 3R after the $n$-th shock–junction interaction;
- $x_4^n$ is the value of the flux on road 4R after the $n$-th shock–junction interaction;

- $x_5^n$ is the value of the flux on road 1 after the $n$-th shock–junction interaction;
- $x_6^n$ is the value of the flux on road 2 after the $n$-th shock–junction interaction.

Defining $x^n \in \mathbb{R}^6$ to be the vector with $x_i^n$ as components, the evolution

$$x^{n+1} = A\,x^n \tag{3.8}$$

is obtained with the definition of the vector $x^n \in \mathbb{R}^6$ with $x_i^n$ as components, where

$$A := \begin{pmatrix} 0 & 0 & \frac{1-q_2}{(1-\alpha)q_1} & 0 & 0 & 0 \\ 0 & 0 & 1-q_2 & 0 & 0 & 0 \\ \frac{1-q_1}{(1-\beta)q_2} & 0 & 0 & 0 & 0 & 0 \\ (1-q_1) & 0 & 0 & 0 & 0 & 0 \\ q_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q_2 & 0 & 0 & 0 \end{pmatrix}$$

and it follows that it suffices to consider the evolution of variables $x_1^n$ and $x_3^n$. The corresponding reduced matrix is

$$\widetilde{A} := \begin{pmatrix} \frac{1-q_2}{(1-\alpha)q_1} & 0 \\ 0 & \frac{1-q_1}{(1-\beta)q_2} \end{pmatrix}$$

and the eigenvalues are given by

$$\lambda^2 = \frac{(1-q_1)(1-q_2)}{(1-\alpha)(1-\beta)q_1 q_2}.$$

**Proposition 3.3** *Assume that:*

$$1 - \frac{(1-\beta)f_2}{f(\sigma)} < q_1 < \frac{f_1}{f(\sigma)},$$

$$1 - \frac{(1-\alpha)f_1}{f(\sigma)} < q_1 < \frac{f_2}{f(\sigma)}.$$

*Then the traffic flow does not stop if the following holds:*

$$\frac{(1-q_1)(1-q_2)}{(1-\alpha)(1-\beta)q_1 q_2} > 1. \tag{3.9}$$

Whenever case (c) (strictly) holds for both junctions (1,4R,1R) and (2,2R,3R), then rarefactions are generated on roads 1R and 3R and shocks on roads 2R and 4R. Furthermore we have the inequalities:

$$q_1 f(\sigma) < f_1, \qquad q_2 f(\sigma) < f_2,$$

$$(1-q_1)f(\sigma) < (1-\alpha)f_1, \qquad (1-q_2)f(\sigma) < (1-\beta)f_2,$$

and it is easy to verify that condition (3.9) can not hold. Hence the following result is established.

**Proposition 3.4** *Assume that:*

$$q_1 > \frac{f_1}{f(\sigma)},$$
$$q_2 > \frac{f_2}{f(\sigma)},$$

*then the circle does get stuck.*

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